

Spinor description of $D = 5$ massless low-spin gauge fields

D.V. Uvarov¹

*NSC Kharkov Institute of Physics and Technology,
61108 Kharkov, Ukraine*

Abstract

Spinor description for the curvatures of $D = 5$ Yang-Mills, Rarita-Schwinger and gravitational fields is elaborated. Restrictions imposed on the curvature spinors by the dynamical equations and Bianchi identities are analyzed. In the absence of sources symmetric curvature spinors with $2s$ indices obey first-order equations that in the linearized limit reduce to Dirac-type equations for massless free fields. These equations allow for a higher-spin generalization similarly to $4d$ case. Their solution in the form of the integral over Lorentz-harmonic variables parametrizing coset manifold $SO(1,4)/(SO(1,1) \times ISO(3))$ isomorphic to the three-sphere is considered. Superparticle model that contains such Lorentz harmonics as dynamical variables, as well as harmonics parametrizing the two-sphere $SU(2)/U(1)$ is proposed. The states in its spectrum are given by the functions on S^3 that upon integrating over the Lorentz harmonics reproduce on-shell symmetric curvature spinors for various massless supermultiplets of $D = 5$ space-time supersymmetry.

1 Introduction

Spinor approach to the description of $4d$ gravitational field initiated by R. Penrose [1] can be extended in the uniform way to other gauge fields that manifests itself in the construction of the contour integral solutions of free massless field equations [2] using the features of two-component $SL(2, \mathbb{C})$ spinors and $SU(2, 2)$ twistors [3]. Symmetric curvature spinors with $2s$ indices that appear in such an approach generalize linearized Weyl curvature spinor ($s = 2$) and were shown later to play an important role in the formulation of higher spin gauge theories extending Einstein [4] and Weyl [5] gravity theories. Generalizations of such a construction relying on twistors related to higher-dimensional conformal symmetries encounter difficulties (see, e.g., [6], [7]) because of the more complicated algebra of multicomponent spinors and the fact that in dimensions greater than four the requirement of conformal invariance turns out very restrictive.

That is why an approach, in which only Lorentz invariance is manifest, looks more preferable. In Ref. [8] there was constructed on-shell integral representation² for the curvatures of massless free fields in dimensions $D = 3, 4, 6, 10$ that uses Lorentz vector and spinor harmonics³ so that the Lorentz-symmetry is built in. The integrand is a function of the rectangular

¹E-mail: d.uvarov@hotmail.com

²Analogous integral representation for solutions of the equations for $4d$ massless arbitrary spin fields had been independently proposed in [9].

³Recall that Lorentz harmonics were introduced in [10] as a generalization of the harmonic variables for compact groups [11]. Spinor harmonics were introduced in [8], [12], [13] (For early attempts on introducing spinor harmonics see, e.g., [14],[15]). Let us note that in $D = 4$ spinor harmonic matrix can be identified with the normalized dyad [16]. The original motivation for considering such objects was connected with the hope to solve the problem of covariant quantization of the Green-Schwarz superstrings using the spinor harmonics. Recently spinor variables that could be related to gauge-fixed Lorentz harmonics were introduced in [17], [18], [19] and used to elaborate on the spinor-helicity formalism for higher-dimensional gauge fields and spinor representation of the scattering amplitudes.

block of the spinor harmonics matrix and the projection of the space-time coordinate vector onto a null vector constructed out of those spinor harmonics. The form of the harmonic measure ensures invariance of the integral under $SO(1, 1) \times ISO(D - 2)$ gauge symmetry so that the integration is actually performed over the sphere S^{D-2} . Resultant curvature spinors are functions of the space-time coordinates and satisfy Dirac-type equations.

In the present paper we generalize this approach to the $5d$ case. To this end we work out in detail the spinor description for $D = 5$ Yang-Mills (YM), Rarita-Schwinger (RS) and gravitational fields. Their irreducible curvature spinors are characterized and contents of the dynamical equations and Bianchi identities written in terms of the curvature spinors is analyzed. It is shown that in the absence of sources there remain non-zero only totally symmetric curvature spinors with $2s$ indices that satisfy first-order differential equations. In the linearized limit they are shown to reduce to the equations for the corresponding linearized curvatures of free massless fields and allow straight-forward higher-spin generalization. These equations may also be viewed as a $5d$ generalization of $4d$ first-order equations [20], [21], [22] obeyed by the generalized Weyl curvature spinors. It should be noted that in the gravitational literature the spinor form of the $D = 5$ Weyl tensor was considered in [23]⁴ and the decomposition of the Riemann tensor on the irreducible curvature spinors was obtained in [25]. Independently in Ref. [26] spinor form of the linearized Weyl tensor and its higher-spin counterparts was used in the unfolded formulation of equations of motions for free fields on AdS_5 that correspond to the unitary irreducible representation of certain higher-spin generalization of $su(2, 2)$ algebra and its supersymmetric extension pertinent to AdS_5/CFT_4 duality.⁵

Then we present integral representation for Weyl curvature spinors of arbitrary spin free fields that makes use of $D = 5$ Lorentz harmonics⁶ and solves Dirac-type equations analogously to the cases considered in Ref. [8].

In the last part of the paper we propose a superparticle model characterized by the set of simple irreducible constraints, whose first-quantized states are given by the multiplets of harmonic functions that correspond to Weyl curvature spinors of the fields from various $D = 5$ supermultiplets. The model includes $D = 5$ spinor harmonics parametrizing the coset $SO(1, 4)/(SO(1, 1) \times ISO(3))$ and harmonics of Ref. [11] parametrizing the two-sphere among the dynamical variables. In the appendices are collected relevant properties of spinors, γ -matrices and Lorentz harmonics.

2 Curvature spinors of $D = 5$ massless gauge fields and equations for them

In this section we find $Spin(1, 4)$ spinor form of the curvatures for YM, RS and gravitational fields, study in detail the restrictions imposed on the curvature spinors by field equations and Bianchi identities and consider their linearized limit.

⁴See also [24], where additionally spinor form of the Maxwell curvature was discussed.

⁵Non-linear equations for symmetric massless higher-spin fields on D -dimensional $(A)dS$ space were constructed in [27] using vectorial generating elements of the underlying higher-spin algebra. We note, however, the importance of spinorial realization of higher-spin algebras and field dynamics in view of possible supersymmetric generalizations (for $D = 5$ case see [28]).

⁶Lorentz harmonics in dimension $D = 5$ were first introduced in [29], although there another coset realization was considered.

2.1 Spinor form of YM field equations and Bianchi identities

We start with the YM case.⁷ YM curvature tensor F_{mn} is converted into the spinor form by contracting vector indices with those of $D = 5$ γ -matrices

$$F_{\alpha[2]\beta[2]} = C_{\alpha_1\beta_1}F_{\alpha_2\beta_2} - C_{\alpha_1\beta_2}F_{\alpha_2\beta_1} - C_{\alpha_2\beta_1}F_{\alpha_1\beta_2} + C_{\alpha_2\beta_2}F_{\alpha_1\beta_1}.^8 \quad (1)$$

Charge conjugation matrix $C_{\alpha\beta}$ that appears on the r.h.s. is antisymmetric (see Appendix A for spinor algebra in 5 dimensions). For real Yang-Mills field symmetric curvature spinor $F_{\alpha\beta} = F_{\beta\alpha}$ satisfies Hermiticity condition

$$(F_{\alpha\beta})^\dagger = \gamma^{0\beta\gamma}F_{\gamma\delta}\gamma^{0\delta\alpha} \quad (2)$$

and thus has 10 real components. Spinor and vector representations of the curvature can be converted into one another using the relations

$$F_{\alpha(2)} = -\frac{1}{2}\gamma^{mn}{}_{\alpha(2)}F_{mn}, \quad F_{mn} = -\frac{1}{4}\gamma_{mn}{}^{\alpha(2)}F_{\alpha(2)}. \quad (3)$$

Vacuum Yang-Mills equations

$$\nabla^m F_{mn} = 0 \quad (4)$$

in spinor representation read

$$\nabla_\alpha{}^\gamma F_{\gamma\beta} - \nabla_\beta{}^\gamma F_{\gamma\alpha} = 0, \quad (5)$$

where $\nabla_\alpha{}^\beta = \gamma^m{}_\alpha{}^\beta \nabla_m$ is the spinor form of the YM covariant derivative. Bianchi identity $\nabla \wedge F = 0$ transforms into equation

$$\nabla_\alpha{}^\gamma F_{\gamma\beta} + \nabla_\beta{}^\gamma F_{\gamma\alpha} = 0. \quad (6)$$

Combining Eqs. (5) and (6) yields that the curvature spinor satisfies

$$\nabla_\alpha{}^\gamma F_{\gamma\beta} = 0. \quad (7)$$

In the linearized limit one obtains free spin-1 field equation

$$\partial_\alpha{}^\gamma F_{\gamma\beta} = 0. \quad (8)$$

2.2 Spinor form of RS field equations and Bianchi identities

In general the RS field $\Psi_{m\gamma}{}^a$ carries Lorentz vector and spinor indices, we additionally endow it with the index a to account for the case of gravitini fields in N -extended supergravity multiplets. Curvature spin-tensor

$$\Phi_{mn\gamma}{}^a = \partial_m \Psi_{n\gamma}{}^a - \partial_n \Psi_{m\gamma}{}^a \quad (9)$$

similarly to the YM curvature tensor can be converted to the spinor form

$$\Phi_{\alpha[2]\beta[2]\gamma}{}^a = C_{\alpha_1\beta_1}\Phi_{\alpha_2\beta_2|\gamma}{}^a - C_{\alpha_1\beta_2}\Phi_{\alpha_2\beta_1|\gamma}{}^a - C_{\alpha_2\beta_1}\Phi_{\alpha_1\beta_2|\gamma}{}^a + C_{\alpha_2\beta_2}\Phi_{\alpha_1\beta_1|\gamma}{}^a. \quad (10)$$

⁷For simplicity we suppress the gauge algebra indices.

⁸For the spinor indices we adopt the shorthand notation that a number in square brackets following an index stands for the group of indices equal to that number that are antisymmetrized with unit weight. Similarly a number in round brackets following an index denotes the group of indices symmetrized with unit weight.

Curvature spinor

$$\Phi_{\alpha(2)|\beta}{}^a = \frac{1}{2}(\partial_{\alpha_1}{}^\delta \Psi_{\alpha_2\delta|\beta}{}^a + \partial_{\alpha_2}{}^\delta \Psi_{\alpha_1\delta|\beta}{}^a), \quad \Psi_{\alpha[2]|\beta}{}^a = \gamma^m{}_{\alpha_1\alpha_2} \Psi_{m\beta}{}^a \quad (11)$$

is symmetric in the first two indices that are therefore separated from the last one by vertical line and satisfies Hermiticity condition

$$(\Phi_{\alpha\beta|\gamma}{}^a)^\dagger = \Omega_{ab} \gamma^{0\alpha\lambda} \gamma^{0\beta\mu} \Phi_{\lambda\mu|\rho}{}^b \gamma^{0\rho\gamma}, \quad (12)$$

where $\Omega_{ab} = -\Omega_{ba}$ is the symplectic metric tensor.⁹ It has $40N$ components and can be presented as the sum of two summands having $20N$ components each

$$\Phi_{\alpha(2)|\beta}{}^a = \Phi_{\alpha_1\alpha_2\beta}{}^a + \widehat{\Phi}_{\alpha(2)|\beta}{}^a. \quad (13)$$

The first term is totally symmetric in the spinor indices

$$\Phi_{\alpha(3)}{}^a = \frac{1}{3} (\Phi_{\alpha_1\alpha_2|\alpha_3}{}^a + \Phi_{\alpha_2\alpha_3|\alpha_1}{}^a + \Phi_{\alpha_3\alpha_1|\alpha_2}{}^a), \quad (14)$$

while the second

$$\widehat{\Phi}_{\alpha(2)|\beta}{}^a = \frac{2}{3} \left(\Phi_{\alpha_1\alpha_2|\beta}{}^a - \frac{1}{2} \Phi_{\alpha_2\beta|\alpha_1}{}^a - \frac{1}{2} \Phi_{\beta\alpha_1|\alpha_2}{}^a \right) \quad (15)$$

has the same symmetry as $\Phi_{\alpha(2)|\beta}{}^a$ but its totally symmetrized part vanishes

$$\widehat{\Phi}_{\alpha_1\alpha_2|\beta}{}^a + \widehat{\Phi}_{\alpha_2\beta|\alpha_1}{}^a + \widehat{\Phi}_{\beta\alpha_1|\alpha_2}{}^a = 0. \quad (16)$$

RS equation

$$\gamma^{klm}{}_\alpha{}^\beta \Phi_{lm\beta}{}^a = 0 \quad (17)$$

transforms into the following equation for $\widehat{\Phi}_{\alpha(2)|\beta}{}^a$:

$$\widehat{\Phi}_{\alpha_1\alpha_2|\beta}{}^a - \widehat{\Phi}_{\alpha_1\beta|\alpha_2}{}^a = 0. \quad (18)$$

Combined with (16) it amounts to

$$\widehat{\Phi}_{\alpha(2)|\beta}{}^a = 0. \quad (19)$$

Bianchi identity for the curvature spin-tensor (9) in the spinor form reads

$$\varrho_{\alpha(2)|\beta}{}^a = \partial_{\alpha_1}{}^\delta \Phi_{\delta\alpha_2|\beta}{}^a + \partial_{\alpha_2}{}^\delta \Phi_{\delta\alpha_1|\beta}{}^a = 0. \quad (20)$$

Since its symmetry is the same as that of $\Phi_{\alpha(2)|\beta}{}^a$, decomposition analogous to (13) applies also to $\varrho_{\alpha(2)|\beta}{}^a$. Provided RS equation (19) is satisfied, the totally symmetric part $\varrho_{\alpha(3)}{}^a$ is expressed in terms of the totally symmetric part of the curvature spinor (14)

$$\varrho_{\alpha(3)}{}^a = \frac{2}{3} (\partial_{\alpha_1}{}^\delta \Phi_{\delta\alpha_2\alpha_3}{}^a + \partial_{\alpha_2}{}^\delta \Phi_{\delta\alpha_3\alpha_1}{}^a + \partial_{\alpha_3}{}^\delta \Phi_{\delta\alpha_1\alpha_2}{}^a) = 0. \quad (21)$$

For another part $\widehat{\varrho}_{\alpha(2)|\beta}{}^a$, whose total symmetrization identically vanishes, one obtains

$$\widehat{\varrho}_{\alpha(2)|\beta}{}^a = -\frac{2}{3} \left(\partial_\beta{}^\delta \Phi_{\delta\alpha_1\alpha_2}{}^a - \frac{1}{2} \partial_{\alpha_1}{}^\delta \Phi_{\delta\alpha_2\beta}{}^a - \frac{1}{2} \partial_{\alpha_2}{}^\delta \Phi_{\delta\alpha_1\beta}{}^a \right) = 0. \quad (22)$$

If further take into account the following representation for the derivative of the totally symmetric part of the curvature (14)

$$\partial_\alpha{}^\delta \Phi_{\delta\beta_1\beta_2}{}^a = \frac{1}{2} \varrho_{\alpha\beta_1\beta_2}{}^a - \widehat{\varrho}_{\beta_1\beta_2|\alpha}{}^a, \quad (23)$$

we conclude that

$$\partial_\alpha{}^\delta \Phi_{\delta\beta(2)}{}^a = 0, \quad (24)$$

whenever RS equation and Bianchi identities are fulfilled.

⁹Recall that allowed R -symmetry groups in $D = 5$ supergravities are $USp(N)$ with $N = 2, 4, 6, 8$.

2.3 Spinor form of gravitational field equations and Bianchi identities

This section culminates in a discussion of $D = 5$ Einstein gravity. Spinor form of the Riemann tensor

$$\begin{aligned}
R_{\alpha[2]\beta[2]\gamma[2]\delta[2]} = & \\
& + C_{\alpha_1\beta_1} (C_{\gamma_1\delta_1} R_{\alpha_2\beta_2|\gamma_2\delta_2} - C_{\gamma_1\delta_2} R_{\alpha_2\beta_2|\gamma_2\delta_1} - C_{\gamma_2\delta_1} R_{\alpha_2\beta_2|\gamma_1\delta_2} + C_{\gamma_2\delta_2} R_{\alpha_2\beta_2|\gamma_1\delta_1}) \\
& - C_{\alpha_1\beta_2} (C_{\gamma_1\delta_1} R_{\alpha_2\beta_1|\gamma_2\delta_2} - C_{\gamma_1\delta_2} R_{\alpha_2\beta_1|\gamma_2\delta_1} - C_{\gamma_2\delta_1} R_{\alpha_2\beta_1|\gamma_1\delta_2} + C_{\gamma_2\delta_2} R_{\alpha_2\beta_1|\gamma_1\delta_1}) \\
& - C_{\alpha_2\beta_1} (C_{\gamma_1\delta_1} R_{\alpha_1\beta_2|\gamma_2\delta_2} - C_{\gamma_1\delta_2} R_{\alpha_1\beta_2|\gamma_2\delta_1} - C_{\gamma_2\delta_1} R_{\alpha_1\beta_2|\gamma_1\delta_2} + C_{\gamma_2\delta_2} R_{\alpha_1\beta_2|\gamma_1\delta_1}) \\
& + C_{\alpha_2\beta_2} (C_{\gamma_1\delta_1} R_{\alpha_1\beta_1|\gamma_2\delta_2} - C_{\gamma_1\delta_2} R_{\alpha_1\beta_1|\gamma_2\delta_1} - C_{\gamma_2\delta_1} R_{\alpha_1\beta_1|\gamma_1\delta_2} + C_{\gamma_2\delta_2} R_{\alpha_1\beta_1|\gamma_1\delta_1})
\end{aligned} \tag{25}$$

can be viewed as the 'square' of the corresponding spinor form of the YM curvature (1). Riemann curvature spinor $R_{\alpha(2)|\beta(2)}$ is symmetric in the first and the second pairs of indices and under their interchange $R_{\alpha(2)|\beta(2)} = R_{\beta(2)|\alpha(2)}$. Inverse relation gives the Riemann tensor in terms of the Riemann spinor

$$R_{klmn} = \frac{1}{16} \gamma_{kl}^{\alpha(2)} \gamma_{mn}^{\beta(2)} R_{\alpha(2)|\beta(2)}. \tag{26}$$

Taking trace in the two vector indices in (26) expresses Ricci tensor via the Riemann curvature spinor

$$R_{km} = R_{kpm}{}^p = -\frac{1}{8} \gamma_k^{\alpha_1\beta_1} \gamma_m^{\alpha_2\beta_2} R_{\alpha_1\alpha_2|\beta_1\beta_2} + \frac{1}{8} \eta_{km} R_{\alpha\beta}{}^{\alpha\beta}. \tag{27}$$

Thus the spinor form of the Ricci tensor is

$$\begin{aligned}
R_{\alpha[2]\beta[2]} = & R_{\alpha_1\beta_2|\alpha_2\beta_1} - R_{\alpha_1\beta_1|\alpha_2\beta_2} \\
& - \frac{1}{2} (C_{\alpha_1\beta_1} R_{\alpha_2\gamma}{}^\gamma{}_{\beta_2} - C_{\alpha_1\beta_2} R_{\alpha_2\gamma}{}^\gamma{}_{\beta_1} - C_{\alpha_2\beta_1} R_{\alpha_1\gamma}{}^\gamma{}_{\beta_2} + C_{\alpha_2\beta_2} R_{\alpha_1\gamma}{}^\gamma{}_{\beta_1}).
\end{aligned} \tag{28}$$

Further tracing gives scalar curvature

$$R = \frac{1}{2} R_{\alpha\beta}{}^{\alpha\beta}. \tag{29}$$

Consider the properties of the Riemann curvature spinor. $R_{\alpha(2)|\beta(2)}$ has 55 components, as the Riemann tensor, and is reducible. It can be represented as the sum

$$R_{\alpha(2)|\beta(2)} = W_{\alpha(2)\beta(2)} + \hat{R}_{\alpha(2)|\beta(2)}. \tag{30}$$

The first summand is the totally symmetric Weyl curvature spinor

$$W_{\alpha(4)} = \frac{1}{3} (R_{\alpha_1\alpha_2|\alpha_3\alpha_4} + R_{\alpha_1\alpha_3|\alpha_4\alpha_2} + R_{\alpha_1\alpha_4|\alpha_2\alpha_3}) \tag{31}$$

and has 35 components, while the second

$$\hat{R}_{\alpha(2)|\beta(2)} = \frac{2}{3} \left(R_{\alpha_1\alpha_2|\beta_1\beta_2} - \frac{1}{2} R_{\alpha_1\beta_1|\beta_2\alpha_2} - \frac{1}{2} R_{\alpha_1\beta_2|\alpha_2\beta_1} \right) \tag{32}$$

has the same symmetries as $R_{\alpha(2)|\beta(2)}$ but its symmetrized part vanishes leaving 20 independent components. $\hat{R}_{\alpha(2)|\beta(2)}$ can be further decomposed into three irreducible spinors

$$\begin{aligned}
\hat{R}_{\alpha(2)|\beta(2)} = & \bar{R}_{\alpha(2)|\beta(2)} - \frac{1}{6} (C_{\alpha_1\beta_1} \tilde{R}_{\alpha_2\beta_2} + C_{\alpha_1\beta_2} \tilde{R}_{\alpha_2\beta_1} + C_{\alpha_2\beta_1} \tilde{R}_{\alpha_1\beta_2} + C_{\alpha_2\beta_2} \tilde{R}_{\alpha_1\beta_1}) \\
& + \frac{1}{20} \hat{R} (C_{\alpha_1\beta_1} C_{\alpha_2\beta_2} + C_{\alpha_1\beta_2} C_{\alpha_2\beta_1}).
\end{aligned} \tag{33}$$

4-index irreducible curvature spinor $\bar{R}_{\alpha(2)|\beta(2)}$ has the same symmetries as $\hat{R}_{\alpha(2)|\beta(2)}$ but additionally all possible traces that one can take using the charge conjugation matrix $C_{\gamma\delta}$ vanish thus leaving 14 independent components. Note that defined in Ref. [25] curvature spinor $\Omega_{\alpha_1\beta_1|\alpha_2\beta_2}$, that is antisymmetric in the first and the second pairs of indices, admits the following expression in terms of $\bar{R}_{\alpha(2)|\beta(2)}$

$$\Omega_{\alpha_1\beta_1|\alpha_2\beta_2} = \bar{R}_{\alpha_1\beta_2|\alpha_2\beta_1} - \bar{R}_{\alpha_1\alpha_2|\beta_1\beta_2}. \quad (34)$$

2-index curvature spinor $\tilde{R}_{\alpha[2]}$ being antisymmetric and traceless has 5 components, while the scalar \hat{R} equals

$$\hat{R} = 2R. \quad (35)$$

Substitution of (33) into (28) allows to express spinor form of the Ricci tensor via the irreducible curvature spinors

$$\begin{aligned} R_{\alpha[2]\beta[2]} &= \bar{R}_{\alpha_1\beta_2|\alpha_2\beta_1} - \bar{R}_{\alpha_1\beta_1|\alpha_2\beta_2} \\ &- \frac{1}{10}\hat{R}(C_{\alpha_1\alpha_2}C_{\beta_1\beta_2} - 2C_{\alpha_1\beta_1}C_{\alpha_2\beta_2} + 2C_{\alpha_1\beta_2}C_{\alpha_2\beta_1}). \end{aligned} \quad (36)$$

The contribution of the 2-index curvature spinor $\tilde{R}_{\alpha[2]}$ drops out because of the relation

$$C_{\beta\gamma}\tilde{R}_{\delta\varepsilon} + C_{\delta\varepsilon}\tilde{R}_{\beta\gamma} + C_{\gamma\delta}\tilde{R}_{\beta\varepsilon} + C_{\beta\varepsilon}\tilde{R}_{\gamma\delta} - C_{\beta\delta}\tilde{R}_{\gamma\varepsilon} - C_{\gamma\varepsilon}\tilde{R}_{\beta\delta} = 0 \quad (37)$$

that can be derived from the identity

$$\tilde{R}_{\lambda[\alpha\varepsilon\beta\gamma\delta\varepsilon]} = 0 \quad (38)$$

upon taking into account the realization of the totally antisymmetric 4-index spinor in terms of the charge conjugation matrices

$$\varepsilon_{\alpha\beta\gamma\delta} = -C_{\alpha\beta}C_{\gamma\delta} + C_{\alpha\gamma}C_{\beta\delta} - C_{\alpha\delta}C_{\beta\gamma} \quad (39)$$

and tracelessness of $\tilde{R}_{\alpha[2]}$. In fact it can be shown that $\tilde{R}_{\alpha[2]}$ vanishes since the algebraic Bianchi identity

$$R_{[klmn]} = 0 \quad (40)$$

in the spinor form amounts to

$$\tilde{R}_{\alpha[2]} = 0. \quad (41)$$

From the definition of the Einstein tensor

$$\mathcal{E}_{mn} = R_{mn} - \frac{1}{2}\eta_{mn}R \quad (42)$$

it is possible to express it in terms of the irreducible curvature spinors

$$\mathcal{E}_{\alpha[2]\beta[2]} = \bar{R}_{\alpha_1\beta_2|\alpha_2\beta_1} - \bar{R}_{\alpha_1\beta_1|\alpha_2\beta_2} + \frac{3}{20}\hat{R}(C_{\alpha_1\alpha_2}C_{\beta_1\beta_2} - 2C_{\alpha_1\beta_1}C_{\alpha_2\beta_2} + 2C_{\alpha_1\beta_2}C_{\alpha_2\beta_1}). \quad (43)$$

So that vacuum Einstein equations yield $\bar{R}_{\alpha(2)|\beta(2)} = \hat{R} = 0$ leaving Weyl curvature spinor as the only non-vanishing quantity.

We conclude the discussion of $D = 5$ Einstein gravity with the analysis of the spinor form of the second Bianchi identity

$$\mathcal{D}_{[k}R_{lm]pr} = 0. \quad (44)$$

Using the spinor form of the covariant derivative and the relation between the Riemann tensor and curvature spinor (26) one obtains equivalent form of the second Bianchi identity

$$B_{\alpha(2)|\beta(2)} = \mathcal{D}_{\alpha_1}{}^\lambda R_{\lambda\alpha_2|\beta(2)} + \mathcal{D}_{\alpha_2}{}^\lambda R_{\lambda\alpha_1|\beta(2)} = 0. \quad (45)$$

4-index spinor $B_{\alpha(2)|\beta(2)}$ is symmetric in the first and the second pairs of indices. It can be presented as the sum of two 4-index spinors symmetric and antisymmetric under permutation of the pairs of indices

$$B_{\alpha(2)|\beta(2)} = S_{\alpha(2)|\beta(2)} + A_{\alpha(2)|\beta(2)} : \quad (46a)$$

$$S_{\alpha(2)|\beta(2)} = \frac{1}{2} \left(\mathcal{D}_{\alpha_1}{}^\lambda R_{\lambda\alpha_2|\beta(2)} + \mathcal{D}_{\alpha_2}{}^\lambda R_{\lambda\alpha_1|\beta(2)} + \mathcal{D}_{\beta_1}{}^\lambda R_{\lambda\beta_2|\alpha(2)} + \mathcal{D}_{\beta_2}{}^\lambda R_{\lambda\beta_1|\alpha(2)} \right), \quad (46b)$$

$$A_{\alpha(2)|\beta(2)} = \frac{1}{2} \left(\mathcal{D}_{\alpha_1}{}^\lambda R_{\lambda\alpha_2|\beta(2)} + \mathcal{D}_{\alpha_2}{}^\lambda R_{\lambda\alpha_1|\beta(2)} - \mathcal{D}_{\beta_1}{}^\lambda R_{\lambda\beta_2|\alpha(2)} - \mathcal{D}_{\beta_2}{}^\lambda R_{\lambda\beta_1|\alpha(2)} \right). \quad (46c)$$

$S_{\alpha(2)|\beta(2)}$ has exactly the same symmetries as the Riemann curvature spinor. So its decomposition into irreducible spinors parallels that of $R_{\alpha(2)|\beta(2)}$ (cf. Eqs. (30)-(33)). First one singles out totally symmetric part and that, whose symmetrization gives zero,

$$S_{\alpha(2)|\beta(2)} = S_{\alpha(2)\beta(2)} + \hat{S}_{\alpha(2)|\beta(2)}, \quad (47)$$

where

$$S_{\alpha(4)} = \frac{1}{3} (S_{\alpha_1\alpha_2|\alpha_3\alpha_4} + S_{\alpha_1\alpha_3|\alpha_4\alpha_2} + S_{\alpha_1\alpha_4|\alpha_2\alpha_3}) \quad (48)$$

and

$$\hat{S}_{\alpha(2)|\beta(2)} = \frac{2}{3} \left(S_{\alpha_1\alpha_2|\beta_1\beta_2} - \frac{1}{2} S_{\alpha_1\beta_1|\beta_2\alpha_2} - \frac{1}{2} S_{\alpha_1\beta_2|\alpha_2\beta_1} \right). \quad (49)$$

The totally symmetric part is contributed only by the Weyl curvature spinor (31)

$$S_{\alpha(4)} = \frac{1}{2} \left(\mathcal{D}_{\alpha_1}{}^\lambda W_{\lambda\alpha_2\alpha_3\alpha_4} + \mathcal{D}_{\alpha_2}{}^\lambda W_{\lambda\alpha_3\alpha_4\alpha_1} + \mathcal{D}_{\alpha_3}{}^\lambda W_{\lambda\alpha_4\alpha_1\alpha_2} + \mathcal{D}_{\alpha_4}{}^\lambda W_{\lambda\alpha_1\alpha_2\alpha_3} \right), \quad (50)$$

while $\hat{S}_{\alpha(2)|\beta(2)}$ – by $\hat{R}_{\alpha(2)|\beta(2)}$ (32)

$$\hat{S}_{\alpha(2)|\beta(2)} = \frac{1}{2} \left(\mathcal{D}_{\alpha_1}{}^\lambda \hat{R}_{\lambda\alpha_2|\beta(2)} + \mathcal{D}_{\alpha_2}{}^\lambda \hat{R}_{\lambda\alpha_1|\beta(2)} + \mathcal{D}_{\beta_1}{}^\lambda \hat{R}_{\lambda\beta_2|\alpha(2)} + \mathcal{D}_{\beta_2}{}^\lambda \hat{R}_{\lambda\beta_1|\alpha(2)} \right). \quad (51)$$

For future reference let us adduce the decomposition of the covariant derivative of the Weyl curvature spinor

$$\mathcal{D}_\alpha{}^\lambda W_{\lambda\beta(3)} = \frac{1}{2} S_{\alpha\beta(3)} + \mathcal{W}_{\alpha|\beta(3)}, \quad (52)$$

where

$$\mathcal{W}_{\alpha|\beta(3)} = \frac{3}{4} \left(\mathcal{D}_\alpha{}^\lambda W_{\lambda\beta_1\beta_2\beta_3} - \frac{1}{3} \mathcal{D}_{\beta_1}{}^\lambda W_{\lambda\beta_2\beta_3\alpha} - \frac{1}{3} \mathcal{D}_{\beta_2}{}^\lambda W_{\lambda\beta_3\alpha\beta_1} - \frac{1}{3} \mathcal{D}_{\beta_3}{}^\lambda W_{\lambda\alpha\beta_1\beta_2} \right) \quad (53)$$

is symmetric in the last three indices but its symmetrization over all the four indices gives zero. $\hat{S}_{\alpha(2)|\beta(2)}$ analogously to (33) can be represented as the sum

$$\begin{aligned} \hat{S}_{\alpha(2)|\beta(2)} &= \bar{S}_{\alpha(2)|\beta(2)} - \frac{1}{6} \left(C_{\alpha_1\beta_1} \tilde{S}_{\alpha_2\beta_2} + C_{\alpha_1\beta_2} \tilde{S}_{\alpha_2\beta_1} + C_{\alpha_2\beta_1} \tilde{S}_{\alpha_1\beta_2} + C_{\alpha_2\beta_2} \tilde{S}_{\alpha_1\beta_1} \right) \\ &+ \frac{1}{20} \hat{S} (C_{\alpha_1\beta_1} C_{\alpha_2\beta_2} + C_{\alpha_1\beta_2} C_{\alpha_2\beta_1}). \end{aligned} \quad (54)$$

Irreducible spinors on the r.h.s. have the same symmetries as corresponding curvature spinors on the r.h.s. of (33). Explicit expressions for them via the covariant derivatives of the curvature spinors are found to be

$$\begin{aligned}
\bar{S}_{\alpha(2)|\beta(2)} &= \frac{1}{2} (\mathcal{D}_{\alpha_1}{}^\lambda \bar{R}_{\lambda\alpha_2|\beta(2)} + \mathcal{D}_{\alpha_2}{}^\lambda \bar{R}_{\lambda\alpha_1|\beta(2)} + \mathcal{D}_{\beta_1}{}^\lambda \bar{R}_{\lambda\beta_2|\alpha(2)} + \mathcal{D}_{\beta_2}{}^\lambda \bar{R}_{\lambda\beta_1|\alpha(2)}) \\
&+ \frac{1}{12} \mathcal{D}^{\lambda\mu} (C_{\alpha_1\beta_1} \bar{R}_{\lambda\alpha_2|\mu\beta_2} + C_{\alpha_1\beta_2} \bar{R}_{\lambda\alpha_2|\mu\beta_1} + C_{\alpha_2\beta_1} \bar{R}_{\lambda\alpha_1|\mu\beta_2} + C_{\alpha_2\beta_2} \bar{R}_{\lambda\alpha_1|\mu\beta_1}) = 0, \\
\tilde{S}_{\alpha[2]} &= \frac{1}{2} \mathcal{D}^{\lambda\mu} \bar{R}_{\lambda\alpha_1|\mu\alpha_2} + \frac{3}{10} \mathcal{D}_{\alpha[2]} \hat{R} = 0, \\
\hat{S} &= 0
\end{aligned} \tag{55}$$

modulo terms proportional to the covariant derivative of the 2-index curvature spinor $\mathcal{D}\tilde{R}$ that vanish when the spinor form of the algebraic Bianchi identity (41) is taken into account. Thus symmetric under permutation of the pairs of indices part of the second Bianchi identity (46b) contains 54 non-trivial equations for the curvature spinors. For vacuum space-times there remain only 35 equations given by vanishing of (50).

Spinor $A_{\alpha(2)|\beta(2)}$ defined in (46c) can be decomposed into the traceless part and that contributing to the trace

$$A_{\alpha(2)|\beta(2)} = \bar{A}_{\alpha(2)|\beta(2)} - \frac{1}{6} (C_{\alpha_1\beta_1} A_{\alpha_2\beta_2} + C_{\alpha_1\beta_2} A_{\alpha_2\beta_1} + C_{\alpha_2\beta_1} A_{\alpha_1\beta_2} + C_{\alpha_2\beta_2} A_{\alpha_1\beta_1}). \tag{56}$$

Symmetric 2-index spinor $A_{\alpha(2)}$ vanishes, when the algebraic Bianchi identity is satisfied. For the traceless part we obtain

$$\begin{aligned}
\bar{A}_{\alpha(2)|\beta(2)} &= \frac{1}{2} (\mathcal{W}_{\alpha_1|\alpha_2\beta_1\beta_2} + \mathcal{W}_{\alpha_2|\alpha_1\beta_1\beta_2} - \mathcal{W}_{\beta_1|\beta_2\alpha_1\alpha_2} - \mathcal{W}_{\beta_2|\beta_1\alpha_1\alpha_2}) \\
&+ \frac{1}{2} (\mathcal{D}_{\alpha_1}{}^\lambda \bar{R}_{\lambda\alpha_2|\beta(2)} + \mathcal{D}_{\alpha_2}{}^\lambda \bar{R}_{\lambda\alpha_1|\beta(2)} - \mathcal{D}_{\beta_1}{}^\lambda \bar{R}_{\lambda\beta_2|\alpha(2)} - \mathcal{D}_{\beta_2}{}^\lambda \bar{R}_{\lambda\beta_1|\alpha(2)}) = 0.
\end{aligned} \tag{57}$$

For vacuum space-times it amounts to the vanishing of $\mathcal{W}_{\alpha|\beta(3)}$ so that recalling (52) we come to the following equation for the Weyl curvature spinor

$$\mathcal{D}_\alpha{}^\lambda W_{\lambda\beta(3)} = 0 \tag{58}$$

and its linearization around flat background

$$\partial_\alpha{}^\lambda W_{\lambda\beta(3)} = 0. \tag{59}$$

Eq. (59) is the spin-2 counterpart of Eqs. (8), (24) for spin-1 and spin-3/2 fields. In fact the sequence can be continued to accommodate (symmetric) higher-spin free fields. One can define a higher-spin generalization of the Weyl curvature spinor $W_{\alpha(2s)}$ that should satisfy the equation generalizing (59)

$$\partial_\alpha{}^\lambda W_{\lambda\beta(2s-1)} = 0. \tag{60}$$

This equation encompasses both low- and higher-spin cases and applies to arbitrary (half-)integer $s \geq 1/2$. It can be shown [30] that for higher-spin fields on-shell there remains non-zero only the Weyl tensor similarly to the low-spin ones.

3 On-shell integral representation for Weyl curvature spinors of $D = 5$ massless gauge fields

Equation (60) can be solved using the integral formula that maps functions $\phi_{i(2s)}$ on S^3 that carry $2s$ symmetrized indices of the fundamental representation of $SU(2)$ (and possibly

indices of the R -symmetry group representations) to Weyl curvature spinors of spin- s fields in Minkowski space. The role of mediator is played by the $D = 5$ Lorentz-harmonic spinor variables. Present section aims at giving the details of such a construction being the generalization of that elaborated in Ref. [8] for the string-theoretic dimensions $D = 3, 4, 6, 10$.¹⁰ To this end we start by recapitulating necessary properties of $D = 5$ Lorentz-harmonic variables.

Vector Lorentz harmonics are given by 5×5 matrix $n_m^{\mathbf{m}}$ subject to the constraint

$$n_m^{\mathbf{m}} \eta^{mn} n_n^{\mathbf{n}} = \eta^{\mathbf{mn}}, \quad \eta = \text{diag}(-, +, +, +, +) \quad (61)$$

so that it takes value in the Lorentz group $SO(1, 4)$. Light-face indices are acted upon by the left $SO(1, 4)_L$ rotations, while the bold-face ones transform under the right $SO(1, 4)_R$ group, whose $SO(1, 1) \times ISO(3)$ subgroup will be gauged. Taking the first and, e.g., the last columns of the harmonic matrix allows to define two light-like vectors

$$n_m^{\pm 2} = n_m^{\mathbf{0}} \pm n_m^{\mathbf{5}} : \quad n_m^{\pm 2} n^{m \pm 2} = 0, \quad n_m^{\pm 2} n^{m \mp 2} = -2. \quad (62)$$

This introduces decomposition of the vector harmonic matrix into three blocks

$$n_m^{\mathbf{m}} = (n_m^{\pm 2}, n_m^{\mathbf{I}}), \quad \mathbf{I} = 1, 2, 3 \quad (63)$$

reducing manifest $SO(1, 4)_R$ -covariance down to $SO(1, 1) \times SO(3)$. Above introduced components of the matrix $n_m^{\mathbf{m}}$ transform in the following manner under infinitesimal $SO(1, 4)_R$ rotation with parameters $L^{\mathbf{mn}} = (L^{+2-2}, L^{\pm 2\mathbf{I}}, L^{\mathbf{IJ}})$

$$\begin{aligned} \delta n_m^{\pm 2} &= \pm L^{+2-2} n_m^{\pm 2} + L^{\pm 2\mathbf{I}} n_m^{\mathbf{I}}, \\ \delta n_m^{\mathbf{I}} &= -\frac{1}{2} (L^{+2\mathbf{I}} n_m^{-2} + L^{-2\mathbf{I}} n_m^{+2}) + L^{\mathbf{IJ}} n_m^{\mathbf{J}}. \end{aligned} \quad (64)$$

Due to zero norm any of the vectors (62) can be set proportional to $D = 5$ massless particle's momentum. For instance, n_m^{+2} is invariant under the transformations with parameters $L^{\mathbf{IJ}}$ and $L^{-2\mathbf{I}}$, and covariant under those with the parameter L^{+2-2} in (64) corresponding to the $SO(1, 1) \times ISO(3)$ subgroup of $SO(1, 4)_R$. This subgroup can be gauged, so that the vector harmonic matrix will parametrize the $SO(1, 4)/SO(1, 1) \times ISO(3)$ coset-space isomorphic to the 3-sphere. More explicitly one can write $n_m^{+2} = (q^{+2}, q^{+2} k_{\hat{m}})$ with the Euclidean 4-vector $k_{\hat{m}}$: $k^2 = 1$ parametrizing S^3 .

Like in arbitrary dimension, $SO(1, 4)$ -valued matrix can be presented as the 'square' of the $Spin(1, 4)$ matrix $v^{\alpha\mu}$

$$n_m^{\mathbf{m}} = \frac{1}{4} v^{\alpha\mu} \gamma_{m\alpha}^{\beta} \gamma^{\mathbf{m}}_{\mu}{}^{\nu} v_{\beta\nu}, \quad v_{\beta\nu} = C_{\beta\gamma} C_{\nu\lambda} v^{\gamma\lambda}. \quad (65)$$

In analogy with the vector-harmonic matrix, the light-face index of $v^{\alpha\mu}$ transforms under the spinor representation of $SO(1, 4)_L$ and the bold-face index under that of $SO(1, 4)_R$. Orthonormality conditions (61) are then satisfied by virtue of six harmonicity conditions imposed on the 4×4 spinor-harmonic matrix

$$v^{\alpha\mu} C_{\alpha\beta} v^{\beta\nu} = C^{\mu\nu}, \quad (66)$$

where $C_{\alpha\beta}$ and $C^{\mu\nu}$ are charge conjugation matrices acting on the $SO(1, 4)_L$ and $SO(1, 4)_R$ spinor indices respectively. Harmonicity conditions ensure that $v^{\alpha\mu}$ takes value in $Spin(1, 4)$.

¹⁰In [8] there was also given on-shell integral representation for integer-spin fields based on D -dimensional vector harmonics.

Similarly to the decomposition (63) of the vector harmonic matrix on $SO(1,1) \times SO(3)$ covariant blocks spinor harmonics decompose into the following 4×2 blocks

$$v^{\alpha\mu} = \begin{pmatrix} v^{\alpha+i} \\ v^{\alpha-i} \end{pmatrix}, \quad (67)$$

where i is the $SU(2)$ fundamental representation index. In terms of these 4×2 blocks null components of the vector harmonics $n_m^{\pm 2}$ acquire the form

$$n_m^{\pm 2} = \frac{1}{2} v^{\alpha \pm i} \gamma_{m\alpha}^{\beta} v_{\beta}^{\pm i}. \quad (68)$$

Fulfilment of the zero norm and orthonormality conditions (62) is again by virtue of the harmonicity conditions (66), whose component form is

$$v^{\alpha \pm i} v_{\alpha}^{\pm j} = 0, \quad v^{\alpha \pm i} v_{\alpha}^{\mp j} - i\varepsilon^{ij} = 0. \quad (69)$$

More details on the properties of $D = 5$ spinor harmonics can be found in Appendix B.

Harmonic variables can be used to construct $SO(1,4)_L$ -invariant Cartan 1-forms. Those taking value in the Lie algebra of $SO(1,4)_R$ are defined by the relations

$$\Omega^{mn}(d) = \frac{1}{2} (n_m^{n} dn^{mm} - n_m^{m} dn^{mn}) = \frac{1}{2} v^{\alpha\mu} \gamma^{mn}_{\mu}{}^{\nu} dv_{\alpha\nu}. \quad (70)$$

They further decompose on 4 irreducible components under the $SO(1,1) \times SO(3)$ subgroup of $SO(1,4)_R$. Explicit expressions for them in terms of the spinor harmonics will be used below

$$\begin{aligned} \Omega^{+2-2}(d) &= i(v^{\alpha+i} dv_{\alpha}^{-i} - v^{\alpha-i} dv_{\alpha}^{+i}), \\ \Omega^{\pm 2\mathbf{I}}(d) &= -v^{\alpha \pm i} \tau_{\mathbf{I}}^{\phantom{\mathbf{I}}i}{}^j dv_{\alpha}^{\pm j}, \\ \Omega^{\mathbf{I}\mathbf{J}}(d) &= \frac{1}{2} \varepsilon^{IJK} (v^{\alpha+i} \tau_{\mathbf{I}}^{Kj} dv_{\alpha}^{-j} + v^{\alpha-i} \tau_{\mathbf{I}}^{Kj} dv_{\alpha}^{+j}). \end{aligned} \quad (71)$$

Proceed now to discussion of the integral representation for the Weyl curvature spinors of the gauge fields. It has the following form

$$W_{\alpha(2s)}(x^m) = \int_{S^3} \Omega^{+6} v_{\alpha_1}^{+i_1} \dots v_{\alpha_{2s}}^{+i_{2s}} \phi^{-6-2s}_{i(2s)}(x^{+2}, v^{+}). \quad (72)$$

Harmonic measure is given by the 3-form

$$\Omega^{+6} = \varepsilon^{IJK} \Omega^{+2\mathbf{I}} \wedge \Omega^{+2\mathbf{J}} \wedge \Omega^{+2\mathbf{K}}. \quad (73)$$

It is $ISO(3)$ invariant (corresponding parameters in (64) are $L^{-2\mathbf{I}}$ and $L^{\mathbf{I}\mathbf{J}}$) and $SO(1,1)$ covariant so that the integral is $SO(1,1) \times ISO(3)$ invariant. By appropriate parametrization of harmonics it reduces to the standard measure on S^3 written via angle variables. Integrand

$$\phi^{-6-2s}_{i(2s)}(x^{+2}, v^{+}) \quad (74)$$

transforms homogeneously under $SO(1,1)$ with weight $-6 - 2s$ to compensate contributions of harmonic measure and spinor harmonics and depends on the space-time coordinates only through the projection $x^{+2} = x^m n_m^{+2}$. This ensures that $W_{\alpha(2s)}$ defined by (72) satisfies Eq. (60) for $s > 0$ and $\square W_{\alpha(2s)} = 0$ for $s \geq 0$. Note that the integral representation analogous to that of Eq. (72) can also be constructed using v_{α}^{-i} spinor. In that case homogeneity degree in v_{α}^{-i} of the integrand should be opposite to that in (74).

4 Massless particle model in doubly harmonic superspace

In this section we discuss a toy model of the massless superparticle, whose canonical quantization yields functions (74) with various values of s assembled into multiplets of $D = 5$ Poincare supersymmetry. It is characterized by the following action¹¹

$$\begin{aligned} S &= \int d\tau \mathcal{L}, \\ \mathcal{L} &= p^{-2,0} \dot{x}^{+2,0} + \pi^{+,-a} \dot{\theta}^{-,+}_a + \frac{1}{2} \Omega_\tau^{+2,-2,0} (p^{-2,0} x^{+2,0} + \frac{1}{2} \pi^{+,-a} \theta^{-,+}_a) \\ &+ \Omega_\tau^{+2,0I} y^{-2,0I} + \omega_\tau^{0,0} \pi^{+,-a} \theta^{-,+}_a + \omega_\tau^{0,+2} y^{0,-2} \\ &+ q \Omega_\tau^{+2,-2,0} + \ell \omega_\tau^{0,0}. \end{aligned} \quad (75)$$

Coordinate $x^{+2,0}$ (denoted as x^{+2} in section 3) upon quantization will become the argument of the superparticle's wave-function (cf. (74)), Grassmann odd coordinate $\theta^{-,+}_a$ in addition to $SO(1,1)$ and $U(1)$ weights of unit modulus carries index $a = 1, \dots, N$ labeling $D = 5$ supersymmetries; $p^{-2,0}$ and $\pi^{+,-a}$ are their canonical momenta. Action (75) also contains four non-dynamic bosonic coordinates $y^{-2,0I}$ and $y^{0,-2}$ and two sets of harmonic variables. $Spin(1,4)$ Lorentz harmonics $v^{\alpha\pm,0i}$ enter via the world-line projections of Cartan forms $\Omega_\tau^{+2,-2,0}$ and $\Omega_\tau^{+2,0I}$ (cf. (71)). Due to the $ISO(3)$ gauge symmetry with parameters $L^{0,0IJ}(\tau)$ and $L^{-2,0I}(\tau)$, under which variables entering the superparticle's Lagrangian transform as

$$\delta \Omega_\tau^{+2,-2,0} = 0, \quad \delta \Omega_\tau^{+2,0I} = L^{0,0IJ} \Omega_\tau^{+2,0J}, \quad \delta y^{-2,0I} = L^{0,0IJ} y^{-2,0J} \quad (76)$$

and

$$\begin{aligned} \delta \Omega_\tau^{+2,-2,0} &= L^{-2,0I} \Omega_\tau^{+2,0I}, \quad \delta \Omega_\tau^{+2,0I} = 0, \\ \delta y^{-2,0I} &= -\frac{1}{2} L^{-2,0I} (p^{-2,0} x^{+2,0} + \frac{1}{2} \pi^{+,-a} \theta^{-,+}_a + 2q), \end{aligned} \quad (77)$$

Lorentz harmonics parametrize the $SO(1,4)/ISO(3)$ manifold. The action also depends on another set of harmonics $w^{0,\pm i}$ via the world-line projections of the Cartan forms

$$\omega_\tau^{0,0} = \frac{1}{2} (\dot{w}^{0,+i} w^{0,-i} + \dot{w}^{0,-i} w^{0,+i}), \quad \omega_\tau^{0,+2} = \dot{w}^{0,+i} w^{0,+i}. \quad (78)$$

They are subject to the harmonicity condition

$$w^{0,+i} w^{0,-i} - 1 = 0 \quad (79)$$

that is nothing but the unimodularity condition ensuring that harmonic matrix $(w^{0,+i}, w^{0,-i})$ takes value in the $SU(2)$ group. Due to the gauge symmetry of the action (75) with the parameter $\Lambda^{0,-2}(\tau)$

$$\delta \omega_\tau^{0,0} = \Lambda^{0,-2} \omega_\tau^{0,+2}, \quad \delta \omega_\tau^{0,+2} = 0, \quad \delta y^{0,-2} = -\Lambda^{0,-2} (\pi^{+,-a} \theta^{-,+}_a + \ell), \quad (80)$$

these harmonics parametrize S^2 manifold. Two last summands in (75) are harmonic 1d Wess-Zumino terms. The first depends on the Lorentz harmonics and the second – on the

¹¹In this section, because we introduce harmonics $w^{0,\pm i}$ parametrizing the two-sphere $SU(2)/U(1)$ in addition to the Lorentz harmonics, the notation has to be slightly improved. For any quantity $\mathcal{Q}^{\alpha p, rT}$ Lorentz indices (if any) precede integers p and r separated by comma that denote $SO(1,1)$ and $U(1)$ weights respectively. They are followed by other indices that in the above example have been collectively denoted by T . Thus for the introduced in the previous section spinor Lorentz harmonics (67) and $SO(1,4)_L$ -invariant Cartan forms (71) it is additionally specified that they carry zero weight w.r.t. to the $U(1)$ subgroup of $SU(2)$, namely $v^{\alpha\pm i} \equiv v^{\alpha\pm,0i}$, $\Omega^{+2,-2}(d) \equiv \Omega^{+2,-2,0}(d)$, $\Omega^{\pm 2I}(d) \equiv \Omega^{+2,0I}(d)$ etc.

$SU(2)/U(1)$ harmonics.¹² Values of the numerical coefficients q and ℓ will be determined below in terms of N and the maximal spin in the supermultiplet.

Systematic treatment of gauge symmetries is achieved in the framework of the canonical approach to the discussion of which we now turn. Definition of canonical momenta results in the following primary constraints

$$P^{+2,0I} \approx 0, \quad (81a)$$

$$P^{0,+2} \approx 0; \quad (81b)$$

$$T_{\alpha}^{-,0}{}_i = P_{\alpha}^{-,0}{}_i + \frac{i}{2}(p^{-2,0}x^{+2,0} + \frac{1}{2}\pi^{+,-a}\theta^{-,+}_a + 2q)v_{\alpha}^{-,0}{}_i - y^{-2,0I}\tau^I{}_i{}^j v_{\alpha}^{+,0}{}_j \approx 0, \quad (81c)$$

$$T_{\alpha}^{+,0}{}_i = P_{\alpha}^{+,0}{}_i - \frac{i}{2}(p^{-2,0}x^{+2,0} + \frac{1}{2}\pi^{+,-a}\theta^{-,+}_a + 2q)v_{\alpha}^{+,0}{}_i \approx 0; \quad (81d)$$

$$U^{0,-}{}_i = P^{0,-}{}_i - \frac{1}{2}(\pi^{+,-a}\theta^{-,+}_a + \ell)w^{0,-}{}_i - y^{0,-2}w^{0,+}{}_i \approx 0, \quad (81e)$$

$$U^{0,+}{}_i = P^{0,+}{}_i - \frac{1}{2}(\pi^{+,-a}\theta^{-,+}_a + \ell)w^{0,+}{}_i \approx 0, \quad (81f)$$

where $P^{+2,0I}$ and $P^{0,+2}$ are momenta conjugate to $y^{-2,0I}$ and $y^{0,-2}$, $P_{\alpha}^{\mp,0}{}_i$ are momenta for the Lorentz harmonics $v^{\alpha\pm,0i}$ and $P^{0,\mp}{}_i$ – for the $SU(2)$ harmonics $w^{0,\pm i}$. This set of constraints has to be supplemented by the harmonicity conditions (69) and (79) that in the canonical approach should be treated as weak equalities in the sense of Dirac. Since it is convenient to consider them holding as usual equalities, i.e. in the strong sense according to Dirac, the following technical trick can be applied. It is possible to single out part of the constraints on the harmonic momenta that form with the harmonicity conditions conjugate pairs of the second-class constraints and construct associated Dirac brackets. Remaining constraints on the harmonic momenta take value in the right Lorentz algebra and are called covariant momenta. Pivotal property of such Dirac brackets is that on the subspace of the harmonic phase-space spanned by harmonics themselves and covariant momenta they coincide with the Poisson brackets, while remaining constraints including the harmonicity conditions are then fulfilled in the strong sense.¹³

Consider in more detail 4 + 1 constraints (81e), (81f) and (79) in the sector of $SU(2)$ harmonics. Projecting constraints (81e) and (81f) onto the harmonics and taking their linear combinations it is easy to find that the constraint

$$w^{0,+i}U^{0,-}{}_i + w^{0,-i}U^{0,+}{}_i = w^{0,+i}P^{0,-}{}_i + w^{0,-i}P^{0,+}{}_i \approx 0 \quad (82)$$

forms with the harmonicity condition (79) the pair of the second-class constraints that can be converted into strong equalities by introducing Dirac brackets. So that in the sector of $SU(2)$ harmonics there remain three constraints corresponding to covariant momenta generating the $su(2)_R$ algebra

$$\mathcal{R}^{0,0} = w^{0,+i}U^{0,-}{}_i - w^{0,-i}U^{0,+}{}_i = w^{0,+i}P^{0,-}{}_i - w^{0,-i}P^{0,+}{}_i + \theta^{-,+}_a \pi^{+,-a} - \ell \approx 0, \quad (83a)$$

$$\mathcal{R}^{0,+2} = w^{0,+i}U^{0,+}{}_i = w^{0,+i}P^{0,+}{}_i \approx 0, \quad (83b)$$

$$\mathcal{R}^{0,-2} = w^{0,-i}U^{0,-}{}_i = w^{0,-i}P^{0,-}{}_i + y^{0,-2} \approx 0. \quad (83c)$$

¹²Superparticle models with Wess-Zumino terms constructed out the $D = 4$ spinor harmonics were considered in [31], [9] and those constructed out of the $SU(2)$ harmonics in [32].

¹³For detailed discussion of the Hamiltonian description of the Lorentz harmonics see, e.g. [33].

In view of the constraints (81b) and (83c) canonical pair $(P^{0,+2}, y^{0,-2})$ can be excluded from the consideration and in the sector of $SU(2)$ harmonics there remain just two constraints (83a) and (83b). The latter is the generator of the gauge symmetry (80).

In the sector of Lorentz harmonics there are 16+6 constraints (81c), (81d) and (69). Like in the case of $SU(2)$ harmonics, one can single out six $SO(1,4)_L$ -invariant constraints from (81c), (81d) that form conjugate pairs of the second-class constraints with the harmonicity conditions (69). 10 remaining constraints constitute covariant momenta taking value in the $so(1,4)_R$ algebra

$$\begin{aligned}\mathcal{P}^{+2,-2,0} &= 2(v^{\alpha-,0i}T_{\alpha}^{+,0}{}_i - v^{\alpha+,0i}T_{\alpha}^{-,0}{}_i) \\ &= -2(v^{\alpha+,0i}P_{\alpha}^{-,0}{}_i - v^{\alpha-,0i}P_{\alpha}^{+,0}{}_i) \\ &+ 2x^{+2,0}p^{-2,0} - \theta^{-,+}_a\pi^{+,-a} + 4q \approx 0,\end{aligned}\tag{84a}$$

$$\mathcal{P}^{+2,0I} = -2iv^{\alpha+,0i}\tau^I{}_i{}^jT_{\alpha}^{+,0}{}_j = -2iv^{\alpha+,0i}\tau^I{}_i{}^jP_{\alpha}^{+,0}{}_j \approx 0,\tag{84b}$$

$$\mathcal{P}^{-2,0I} = 2iv^{\alpha-,0i}\tau^I{}_i{}^jT_{\alpha}^{-,0}{}_j = 2iv^{\alpha-,0i}\tau^I{}_i{}^jP_{\alpha}^{-,0}{}_j - 4y^{-2,0I} \approx 0,\tag{84c}$$

$$\begin{aligned}\mathcal{P}^{0,0IJ} &= i\varepsilon^{IJK}(v^{\alpha+,0i}\tau^K{}_i{}^jT_{\alpha}^{-,0}{}_j + v^{\alpha-,0i}\tau^K{}_i{}^jT_{\alpha}^{+,0}{}_j) \\ &= i\varepsilon^{IJK}(v^{\alpha+,0i}\tau^K{}_i{}^jP_{\alpha}^{-,0}{}_j + v^{\alpha-,0i}\tau^K{}_i{}^jP_{\alpha}^{+,0}{}_j) \approx 0.\end{aligned}\tag{84d}$$

One observes that due to the constraints (81a) and (84c) canonical variables $(P^{+2,0I}, y^{-2,0I})$ can be excluded from consideration. So that in the sector of Lorentz harmonics one is left with seven constraints (84a), (84b), (84d). Constraints (84d) and (84b) are the generators of the discussed above gauge transformations (76) and (77) respectively.

Thus the constraints (83a), (83b) and (84a), (84b), (84d) form the set of the first-class constraints of our model. Associated quantum operators are imposed on the superparticle's wave function that in the coordinate representation depends on $x^{+2,0}$, $v^{\alpha\pm,0i}$, $w^{0,\pm i}$ and $\theta^{-,+}_a$. Conjugate momenta then act as differential operators

$$\hat{p}^{-2,0} = \frac{\partial}{\partial x^{+2,0}}, \quad \hat{P}_{\alpha}^{\pm,0}{}_i = \frac{\partial}{\partial v^{\alpha\mp,0i}}, \quad \hat{P}^{0,\pm}{}_i = \frac{\partial}{\partial w^{0,\mp i}}, \quad \hat{\pi}^{+,-a} = \frac{\partial}{\partial \theta^{-,+}_a}\tag{85}$$

so that the (anti)commutations relations hold

$$[\hat{p}^{-2,0}, \hat{x}^{+2,0}] = 1, \quad [\hat{P}_{\alpha}^{\pm,0}{}_i, \hat{v}^{\beta\mp,0j}] = \delta_{\alpha}^{\beta}\delta_i^j, \quad [\hat{P}^{0,\pm}{}_i, \hat{w}^{0,\mp j}] = \delta_i^j, \quad \{\hat{\pi}^{+,-a}, \hat{\theta}^{-,+}_b\} = \delta_b^a.\tag{86}$$

Classical generator of the $SO(1,1)$ gauge symmetry (84a) in such a realization transforms into the following Hermitian operator

$$\hat{\Delta}_{so(1,1)} = 2x^{+2,0}\frac{\partial}{\partial x^{+2,0}} + v^{\alpha+,0i}\frac{\partial}{\partial v^{\alpha+,0i}} - v^{\alpha-,0i}\frac{\partial}{\partial v^{\alpha-,0i}} - \theta^{-,+}_a\frac{\partial}{\partial \theta^{-,+}_a} - c_{so(1,1)},\tag{87}$$

where $c_{so(1,1)} = -1 - 4q - \frac{N}{2}$ is an ordering constant. It is nothing but the dilatation operator acting on the variables with non-zero $SO(1,1)$ weights. Similarly quantum $U(1)$ generator associated with the constraint (83a) can be brought to the form

$$\hat{\Delta}_{u(1)} = w^{0,+i}\frac{\partial}{\partial w^{0,+i}} - w^{0,-i}\frac{\partial}{\partial w^{0,-i}} + \theta^{-,+}_a\frac{\partial}{\partial \theta^{-,+}_a} - c_{u(1)},\tag{88}$$

where $c_{u(1)} = \ell + \frac{N}{2}$. It 'measures' the $U(1)$ charges of the components of superparticle's wave function. Quantum operators corresponding to other constraints (84b), (84d) and (83b) are

free from the ordering ambiguities and can be defined in the following way

$$\hat{\mathbf{P}}^{+2,0I} = v^{\alpha+,0i} \tau_i^{I,j} \frac{\partial}{\partial v^{\alpha-,0j}}. \quad (89a)$$

$$\hat{\mathbf{P}}^{0,0IJ} = \varepsilon^{IJK} \left(v^{\alpha+,0i} \tau_i^{K,j} \frac{\partial}{\partial v^{\alpha+,0j}} + v^{\alpha-,0i} \tau_i^{K,j} \frac{\partial}{\partial v^{\alpha-,0j}} \right), \quad (89b)$$

$$\hat{\mathbf{R}}^{0,+2} = w^{0,+i} \frac{\partial}{\partial w^{0,-i}}. \quad (89c)$$

These operators act on the wave function that admits series expansion in θ

$$\begin{aligned} \Phi^{c_{so(1,1)}, c_{u(1)}}(x^{+2,0}, v^{\alpha\pm,0i}, w^{0,\pm i}, \theta^{-,+}_a) = & \varphi^{c_{so(1,1)}, c_{u(1)}} + \theta^{-,+}_a \varphi^{c_{so(1,1)}+1, c_{u(1)}-1 a} \\ & + \dots + \theta^{-k,+k}_{a[k]} \varphi^{c_{so(1,1)}+k, c_{u(1)}-k a[k]} \\ & + \dots + \theta^{-N,+N}_{a[N]} \varphi^{c_{so(1,1)}+N, c_{u(1)}-N a[N]}, \end{aligned} \quad (90)$$

where $\theta^{-k,+k}_{a[k]} \equiv \theta^{-,+}_{a_1} \theta^{-,+}_{a_2} \dots \theta^{-,+}_{a_k}$ and the component functions depend on $x^{+2,0}$ and harmonics. In particular, one finds that for any $0 \leq k \leq N$

$$\hat{\mathbf{P}}^{+2,0I} \varphi^{c_{so(1,1)}+k, c_{u(1)}-k a[k]} = 0, \quad (91a)$$

$$\hat{\mathbf{R}}^{0,+2} \varphi^{c_{so(1,1)}+k, c_{u(1)}-k a[k]} = 0 \quad (91b)$$

and the operator (89b) determines how the wave function components transform under the $SO(3) \sim SU(2)$. The first equation implies that the wave function is independent of $v^{\alpha-,0i}$ (cf. [8]) and the second allows to factorize the dependence on the $SU(2)$ harmonics. Namely, for any function with non-negative value of the $U(1)$ weight $c_{u(1)} - k \geq 0$ it was proved in [11] that the solution of Eq. (91b) is

$$\varphi^{c_{so(1,1)}+k, c_{u(1)}-k a[k]}(x^{+2,0}, v^{\alpha+,0i}, w^{0,+i}) = \varphi^{c_{so(1,1)}+k, 0 a[k]}_{i(c_{u(1)}-k)} w^{0,+i(c_{u(1)}-k)}, \quad (92)$$

where $w^{0,+i(c_{u(1)}-k)} \equiv w^{0,+i_1} w^{0,+i_2} \dots w^{0,+i_{c_{u(1)}-k}}$, while if the $U(1)$ weight is negative the function $\varphi^{c_{so(1,1)}+k, c_{u(1)}-k a[k]}$ vanishes. As a result component functions $\varphi^{c_{so(1,1)}+k, 0 a[k]}_{i(c_{u(1)}-k)}$ depend on $x^{+2,0}$ and $v^{\alpha+,0i}$ only. In order to identify them with the integrand in (72) the following relation between the ordering constants in (87) and (88) should hold

$$c_{so(1,1)} = -6 - c_{u(1)} \quad (93)$$

and then $c_{u(1)} - k = 2s_k$, where s_k is the spin of the corresponding gauge field. Using that the field with the highest value of spin in the multiplet s_{max} corresponds to the leading component in the expansion (90), allows to express coefficients at the Wess-Zumino terms q and ℓ through s_{max} .

Consider in more detail the case of $N = 2$ supersymmetry. From equations

$$c_{so(1,1)} = -6 - 2s_{max}, \quad c_{u(1)} = 2s_{max} \quad (94)$$

it follows that

$$q = 1 + \frac{s_{max}}{2}, \quad \ell = 2s_{max} - 1. \quad (95)$$

If the value of the maximal spin in the supermultiplet is set to $1/2$ we get two non-zero components of the wave function $\varphi^{-7,1}(x^{+2,0}, v^{\alpha+,0i})$ and $\varphi^{-6,0 a}(x^{+2,0}, v^{\alpha+,0i})$ corresponding to the component fields of $N = 2$ hypermultiplet. For $s_{max} = 1$ one obtains three component

fields on S^3 $\varphi^{-8,2}$, $\varphi^{-7,1a}$ and $\varphi^{-6,0a[2]}$ that correspond to the component fields of $N = 2$ Maxwell supermultiplet. Similarly setting $s_{max} = 2$ yields the on-shell field strengths of $N = 2$ supergravity multiplet $(W_{\alpha(4)}, \Phi_{\alpha(3)}^a, F_{\alpha(2)})(x^m)$ [34]. Choosing other values of the maximal spin gives massless higher-spin multiplets of $D = 5$ $N = 2$ supersymmetry. The model (75) is also capable to describe $D = 5$ $N = 4$ supergravity multiplet for $q = 7/4$ and $\ell = 2$. Corresponding components of the superparticle's wave function and associated fields on the space-time are listed in the Table.

Fields on S^3	Fields on space-time
$\varphi^{-10,4}$	$W_{\alpha(4)}$
$\varphi^{-9,3a}$	$\Phi_{\alpha(3)}^a$
$\varphi^{-8,2a[2]}$	$F_{\alpha(2)}^{a[2]}$
$\varphi^{-7,1a}$	$\Psi_{\alpha a}$
$\varphi^{-6,0}$	S

5 Conclusion

Gauge invariant curvatures of YM and gravitational fields in $5d$ are known to admit equivalent spinor representation similarly to corresponding $4d$ fields. In particular, $D = 5$ Riemann tensor amounts to the set of four irreducible (multi-index) spinors. We analyzed restrictions imposed by the dynamical equations and Bianchi identities on the curvature spinors for YM, gravitational and free massless spin-3/2 fields. In analogy with the $4d$ case in the absence of sources there remain non-zero only symmetric curvature spinors with $2s$ indices given by the Weyl curvature spinor and its lower-spin counterparts. These spinors satisfy first-order equations reducing in the linearized limit to Dirac-type equations that can be written in the uniform way for various spins. This suggests that they are the first members of the sequence of equations for higher-spin Weyl curvature spinors. These equations are solved using the integral representation based on writing the null-momentum as the square of spinors that are blocks of the spinor harmonic matrix parametrizing the coset $SO(1,4)/(SO(1,1) \times ISO(3))$ being the realization of the S^3 manifold and the integration is actually performed over this three-sphere. This integral representation for on-shell Weyl curvature spinors is the $D = 5$ extension of that in dimensions $D = 3, 4, 6, 10$ described in [8]. The possibility to elaborate such an integral representation for Weyl curvature spinors in various dimensions and for fields of various spins is due to fact that only Lorentz symmetry is manifest. So it would be interesting to look for generalizations to the fields over curved backgrounds, particularly such as $(A)dS$ ones, and in other dimensions including the string/M-theoretic ones. Less straight-forward but potentially more promising is to promote proposed superparticle model to the string one, whose correlation functions would reproduce (tree-level) scattering amplitudes in $D = 5$ YM/gravity theories similarly to the (ambi)twistor-string models [35], [36], [37], [38], [39]. The fact that spinorial (vectorial) constituents are Lorentz harmonics makes feasible also a generalization to other dimensions like in the case of the ambitwistor string of Ref. [38], in which Lorentz harmonics enter implicitly [40].

Acknowledgements

The author is obliged to A.A. Zheltukhin for stimulating discussions.

Appendix A. γ -matrices and spinors

γ -matrices $\gamma^m_{\alpha}{}^{\beta}$ ($m = 0, 1, 2, 3, 5$, $\alpha, \beta = 1, 2, 3, 4$) satisfy the defining relations of the $D = 5$ Clifford algebra

$$\gamma^m_{\alpha}{}^{\delta} \gamma^n_{\delta}{}^{\beta} + \gamma^n_{\alpha}{}^{\delta} \gamma^m_{\delta}{}^{\beta} = 2\eta^{mn} \delta_{\alpha}^{\beta}. \quad (96)$$

Positions of the spinor indices can be changed with the aid of the charge conjugation matrices $C^{\alpha\beta}$ and $C_{\alpha\beta}$: $C^{\alpha\delta} C_{\delta\beta} = \delta_{\beta}^{\alpha}$ according to the rule

$$\psi^{\alpha} = C^{\alpha\beta} \psi_{\beta}, \quad \chi_{\alpha} = C_{\alpha\beta} \chi^{\beta}. \quad (97)$$

Both charge conjugation and γ -matrices are antisymmetric in 5 dimensions.

Subsequent relations are widely used in the main text

$$(C^{\alpha\beta})^* = -C_{\alpha\beta}, \quad (\gamma^m)^{\dagger} = \gamma^0 \gamma^m \gamma^0, \quad (\gamma^m_{\alpha}{}^{\beta})^T = -\gamma^{m\beta}{}_{\alpha}, \quad (98)$$

where $\gamma^{m\beta}{}_{\alpha} = -C^{\beta\gamma} \gamma^m_{\gamma}{}^{\delta} C_{\delta\alpha}$ and minus sign reflects antisymmetry of the charge conjugation matrices.

In discussion of the Lorentz harmonics used is the light-cone basis for γ -matrices, in which only $SO(1, 1) \times SU(2)$ covariance is manifest. Let 0 and 5 be the light-cone directions, then γ -matrices exhibit the direct product structure

$$\gamma^0_{\mu}{}^{\nu} = \rho^{\dagger}_{(\mu)}{}^{(\nu)} \delta_i^j, \quad \gamma^I_{\mu}{}^{\nu} = \rho^{\text{ts}}_{(\mu)}{}^{(\nu)} \tau^I_i{}^j, \quad \gamma^5 = \rho^s_{(\mu)}{}^{(\nu)} \delta_i^j. \quad (99)$$

Non-relativistic Pauli matrices satisfy

$$\tau^I_i{}^k \tau^J_k{}^j + \tau^J_i{}^k \tau^I_k{}^j = 2\delta^{IJ} \delta_i^j. \quad (100)$$

ρ^{\dagger} and ρ^s are timelike and spacelike γ -matrices in $D = 1 + 1$ dimensions

$$\rho^{\dagger}_{(\mu)}{}^{(\nu)} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \rho^s_{(\mu)}{}^{(\nu)} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad (101)$$

and $\rho^{\text{ts}} = \rho^{\dagger} \rho^s$. Similarly charge conjugation matrices acquire the direct product form

$$C^{\mu\nu} = C^{(\mu)(\nu)} \varepsilon^{ij}, \quad C_{\mu\nu} = C_{(\mu)(\nu)} \varepsilon_{ij}, \quad (102)$$

where antisymmetric unit matrices ε^{ij} and ε_{ij} :

$$\varepsilon^{12} = \varepsilon_{21} = 1, \quad (\varepsilon^{ij})^{\dagger} = \varepsilon_{ji}, \quad \varepsilon^{ij} \varepsilon_{jk} = \delta_k^i \quad (103)$$

are used to move $SU(2)$ indices as $\psi^i = \varepsilon^{ij} \psi_j$, $\chi_i = \varepsilon_{ij} \chi^j$, and $D = 1 + 1$ charge conjugation matrices equal

$$C^{(\mu)(\nu)} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad C_{(\mu)(\nu)} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad (104)$$

so that

$$\rho^{\dagger}_{(\mu)}{}^{(\nu)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho^s_{(\mu)}{}^{(\nu)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (105)$$

as desired for expressing light-cone components of the vector Lorentz harmonics in terms of the spinor ones.

Appendix B. Some properties of $D = 5$ Lorentz harmonics

Discussion of the properties of the spinor harmonics $v^{\alpha\mu}$ in this section complements that in the main text.

Spinor harmonics satisfy generalized Majorana condition

$$\bar{v}^{\alpha\mu} = (v_{\beta\nu})^\dagger \gamma^0{}^\alpha{}_\beta \gamma^0{}^\mu{}_\nu = (C^{\alpha\beta} C^{\mu\nu} v_{\beta\nu})^T = (v^{\alpha\mu})^T \quad (106)$$

reducing the number of (real) independent components from 32 to 16. Other forms of this condition are

$$(v_{\alpha\mu})^* = \gamma^0{}^\alpha{}_\beta \gamma^0{}^\mu{}_\nu v^{\beta\nu}, \quad (v^{\alpha\mu})^* = \gamma^0{}^\alpha{}_\beta \gamma^0{}^\mu{}_\nu v_{\beta\nu} \quad (107)$$

can be used to check reality of the vector harmonics realized in terms of the spinor ones (65).

The light-cone basis realization for $D = 5$ γ -matrices implies decomposition of $Spin(1, 4)$ spinor on $SO(1, 1) \times SU(2)$ representations $\mathbf{4} = \mathbf{2}_+ \oplus \mathbf{2}_-$. Applied to the index that transforms under $SO(1, 4)_R$ introduces decomposition of the spinor harmonic matrix on 4×2 blocks (cf. Eq. (67))

$$v^{\alpha\mu} = v^{\alpha(\mu)i} = \begin{pmatrix} v^{\alpha+i} \\ v^{\alpha-i} \end{pmatrix}, \quad v_{(\mu)i} = C_{(\mu)(\nu)} \varepsilon_{ij} v^{\alpha(\nu)j} = -i \begin{pmatrix} v^{\alpha-}_i \\ v^{\alpha+}_i \end{pmatrix}. \quad (108)$$

For such blocks generalized Majorana condition reduces to the $SU(2)$ -Majorana condition

$$(v^{\alpha\pm i})^* = \gamma^0{}^\alpha{}_\beta v^{\pm}_i. \quad (109)$$

References

- [1] R. Penrose, A spinor approach to general relativity, *Ann. Phys.* **10** (1960) 171.
- [2] R. Penrose, Twistor quantisation and curved space-time, *Int. J. Theor. Phys.* **1** (1968) 61.
- [3] R. Penrose, Twistor algebra, *J. Math. Phys.* **8** (1967) 345.
- [4] M.A. Vasiliev, Consistent equation for interacting gauge fields of all spins in 3+1 dimensions, *Phys. Lett.* **B243** (1990) 378.
- [5] E.S. Fradkin and V.Ya. Linetsky, Cubic interaction in conformal theory of integer higher-spin fields in four-dimensional space-time, *Phys. Lett.* **B231** (1989) 97; Superconformal higher-spin theory in the cubic approximation, *Nucl. Phys.* **B350** (1991) 274.
- [6] C. Saemann and M. Wolf, On twistors and conformal field theories from six dimensions, *J. Math. Phys.* **54** (2013) 013507, arXiv:1111.2539 [hep-th].
- [7] L.J. Mason, R.A. Reid-Edwards and A. Taghavi-Chabert, Conformal field theories in six-dimensional twistor space, *J. Geom. Phys.* **62** (2012) 2353, arXiv:1111.2585 [hep-th].
- [8] F. Delduc, A.S. Galperin and E. Sokatchev, Lorentz-harmonic (super)fields and (super)particles, *Nucl. Phys.* **B368** (1992) 143.

- [9] I.A. Bandos, Superparticle in Lorentz-harmonic superspace, *Sov. J. Nucl. Phys.* **51** (1990) 906 (*Yadern. Fiz.* **51** (1990) 1429).
- [10] E. Sokatchev, Light-cone harmonic superspace and its applications, *Phys. Lett.* **B169** (1986) 209; Harmonic superparticle, *Class. Quantum Grav.* **4** (1987) 237.
- [11] A.S. Galperin, E.A. Ivanov, S. Kalitzin, V.I. Ogievetsky and E. Sokatchev, Unconstrained $N = 2$ matter, Yang-Mills and supergravity theories in harmonic superspace, *Class. Quantum Grav.* **1** (1984) 469.
- [12] A.S. Galperin, P.S. Howe and K.S. Stelle, The superparticle and the Lorentz group, *Nucl. Phys.* **B368** (1992) 248, arXiv:hep-th/9201020.
- [13] I.A. Bandos and A.A. Zheltukhin, Spinor Cartan moving n -hedron, Lorentz-harmonic formulations of superstrings and κ -symmetry, *JETP Lett.* **54** (1991) 421 (*Pis'ma v ZhETF* **54** (1991) 421); Green-Schwarz superstrings in spinor moving frame formalism, *Phys. Lett.* **B288** (1992) 77.
- [14] E. Nissimov, S. Pacheva and S. Solomon, Covariant first and second quantization of the $N = 2$ $D = 10$ Brink-Schwarz superparticle, *Nucl. Phys.* **B296** (1988) 462.
- [15] S. Solomon, New canonical covariant formalism for the $N = 2$ $D = 10$ superparticle, *Phys. Lett.* **B203** (1988) 86.
- [16] E.T. Newman and R. Penrose, An approach to gravitational radiation by a method of spin coefficients, *J. Math. Phys.* **3** (1962) 896.
- [17] C. Cheung and D. O'Connell, Amplitudes and spinor-helicity in six dimensions, *JHEP* **0907** (2009) 075, arXiv:0902.0981 [hep-th].
- [18] R. Boels, Covariant representation theory of the Poincare algebra and some of its extensions, *JHEP* **1002** (2010) 010, arXiv:0908.0738 [hep-th].
- [19] S. Caron-Huot and D. O'Connell, Spinor helicity and dual conformal symmetry in ten dimensions, *JHEP* **1108** (2011) 014, arXiv:1010.5487 [hep-th].
- [20] R. Penrose and M.A.H. MacCallum, Twistor theory: an approach to the quantisation of fields and space-time, *Phys. Rept.* **6** (1972) 241.
- [21] R. Penrose and W. Rindler, *Spinors and space-time. V.1: Two-spinor calculus and relativistic fields*, Cambridge Univ. Press, 1984.
- [22] M.A. Vasiliev, Equations of motion of interacting massless fields of all spins as a free differential algebra, *Phys. Lett.* **B209** (1988) 491.
- [23] P.J. De Smet, Black holes on cylinders are not algebraically special, *Class. Quantum Grav.* **19** (2002) 4877, arXiv:hep-th/0206106.
- [24] M. Godazgar, Spinor classification of the Weyl tensor in five dimensions, *Class. Quantum Grav.* **27** (2010) 245013, arXiv:1008.2995 [gr-qc].
- [25] A. Garcia-Parrado Gomez-Lobo and J.M. Martin-Garcia, Spinor calculus on 5-dimensional spacetimes, *J. Math. Phys.* **50** (2009) 122504, arXiv:0905.2846 [gr-qc].

- [26] E. Sezgin and P. Sundell, Doubletons and $5d$ higher spin gauge theory, JHEP **0109** (2001) 036, arXiv:hep-th/0105001; Towards massless higher spin extension of $D = 5$, $N = 8$ gauged supergravity, JHEP **0109** (2001) 025, arXiv:hep-th/0107186.
- [27] M.A. Vasiliev, Non-linear equations for symmetric massless higher spin fields in $(A)dS_d$, Phys. Lett. **B567** (2003) 139, arXiv:hep-th/0304049.
- [28] M.A. Vasiliev, Cubic interactions of bosonic higher spin gauge fields in AdS_5 , Nucl. Phys. **B616** (2001) 106, arXiv:hep-th/0106200; K.B. Alkalaev and M.A. Vasiliev, $N = 1$ supersymmetric theory of higher spin gauge fields in AdS_5 at the cubic level, Nucl. Phys. **655** (2003) 57, arXiv:hep-th/0206068.
- [29] E. Sokatchev, An off-shell formulation of $N = 4$ supersymmetric Yang-Mills theory in twistor harmonic superspace, Phys. Lett. **B217** (1989) 489.
- [30] V.E. Lopatin and M.A. Vasiliev, Free massless bosonic fields of arbitrary spin in d -dimensional de Sitter space, Mod. Phys. Lett. **A3** (1988) 257; M.A. Vasiliev, Free massless fermionic fields of arbitrary spin in d -dimensional anti-de Sitter space, Nucl. Phys. **B301** (1988) 26.
- [31] I.A. Bandos, Multivalued action functionals, Lorentz harmonics and spin, JETP Lett. **52** (1990) 205 (Pisma v ZhETF **52** (1990) 837).
- [32] V.P. Akulov, I.A. Bandos and D.P. Sorokin, Particle mechanics in harmonic superspace, Mod. Phys. Lett. **A3** (1988) 1633.
- [33] I.A. Bandos and A.A. Zheltukhin, Twistor-like approach in the Green-Schwarz $D = 10$ superstring theory, Phys. Elem. Part. Atom. Nucl. **25** (1994) 453 (EChAYa **25** (1994) 1065).
- [34] E. Cremmer, Supergravities in 5 dimensions, in *Superspace and supergravity*, eds. S.W. Hawking and M. Roček (Cambridge Univ. Press, 1981), pp.267-282.
- [35] E. Witten, Perturbative gauge theory as a string theory in twistor space, Comm. Math. Phys. **252** (2004) 189, arXiv:hep-th/0312171.
- [36] N. Berkovits, Alternative string theory in twistor space for $N = 4$ super-Yang-Mills theory, Phys. Rev. Lett. **93** (2004) 011601, arXiv:hep-th/0402045.
- [37] D. Skinner, Twistor strings for $N = 8$ supergravity, arXiv:1301.0868 [hep-th].
- [38] L. Mason and D. Skinner, Ambitwistor strings and scattering equations, JHEP **1407** (2014) 048, arXiv:1311.2564 [hep-th].
- [39] Y. Geyer, A.E. Lipstein and L. Mason, Ambitwistor strings in four dimensions, Phys. Rev. Lett. **113** (2014) 8, arXiv:1404.6219 [hep-th].
- [40] I. Bandos, Twistor/ambitwistor strings and null-superstrings in space-time of $D = 4, 10$ and 11 dimensions, JHEP **1409** (2014) 086, arXiv:1404.1299 [hep-th].